COMPACT KÄHLER THREEFOLDS WITH NON-NEF CANONICAL BUNDLE AND SYMPLECTIC GEOMETRY

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ABSTRACT. We show that the nefness of the canonical bundle of compact Kähler threefolds is invariant under deformed symplectic diffeomorphisms.

1. INTRODUCTION

Consider a compact Kähler manifold (X, ω) and its canonical bundle K_X. We say that K_X is *algebraically nef* if its degree is non-negative on every curve C on X. We shall briefly call it *nef*[1] throughout this note by abuse of language. Let (Y, ω') be another compact Kähler manifold. A *deformed symplectic diffeomorphism* $\phi : X \to Y$ is a diffeomorphism such that the symplectic form $\phi^*\omega'$ is in the same deformation class of symplectic forms as ω. If such a diffeomorphism ϕ exists, we say that X and Y are *symplectically equivalent*.

A Kähler manifold equipped with its Kähler form is also a symplectic manifold, and furthermore, the set of Kähler forms is connected and thus determines a deformation class of symplectic forms. Hence, it is natural to ask which properties coming from algebraic or Kähler geometry depend only on the symplectic deformation equivalence class. This kind of questions was first studied by Ruan [Rua93, Rua94] by showing that the nefness of the canonical bundle of smooth projective surfaces or threefolds is invariant under deformed symplectic diffeomorphisms. In the same direction, Kollár [Kol98] and Ruan [Rua99] showed that uniruledness of Kähler manifolds is a symplectic invariant. The same holds true for rational connectedness of smooth projective varieties of dimension up to four [Voi08, Tia12a, Tia12b].

The aim of this note is to prove the following Theorem, which generalizes Ruan's result:

Theorem 1.1. *The algebraic nefness of the canonical bundle of compact Kähler threefolds is invariant under deformed symplectic diffeomorphisms: given two symplectically equivalent compact Kähler threefolds X and Y, assume that K_X is nef, then K_Y is also nef.*

Let X be a compact Kähler manifold. Set $N_1(X)_{\mathbb{R}}$[2] the real vector space of 1-cycles modulo numerical equivalence. Inside $N_1(X)_{\mathbb{R}}$ sits the *effective cone*

$$\mathrm{NE}(X) := \left\{ \sum_{\text{finite}} a_i [C_i] \,\middle|\, a_i > 0, C_i \text{ irreducible curves of } X \right\} \subset N_1(X)_{\mathbb{R}},$$

the set of classes of effective 1-cycles. We denote by $\overline{\mathrm{NE}}(X)$ the closure of $\mathrm{NE}(X)$ in $N_1(X)_{\mathbb{R}}$.

More generally, given a symplectic manifold (M, ω) and a ω-tamed almost complex structure J on M, we define

$$\mathrm{NE}(M)_{\omega,J} := \left\{ \sum_{\text{finite}} a_i [C_i] \,\middle|\, a_i > 0, C_i \text{ J-holomorphic curves of } M \right\} \subset H_2(M, \mathbb{R}).$$

In [Rua93], Ruan defined the *deformed symplectic effective cone* as the intersection for all symplectic forms ω' which can deform to ω and all ω'-tamed almost complex structures J

$$\mathrm{DNE}(M) := \bigcap_{\omega'} \bigcap_J \mathrm{NE}(M)_{\omega,J}.$$

In the same paper, he stated the following *main criterion*:

Theorem 1.2. *Suppose that $[C]$ is extremal*[3] *in $\overline{\mathrm{NE}}(M)_{\omega',J}$ for some ω' and J. If the Gromov-Witten invariant $\mathrm{GW}_{[C]}$ is non-zero, then $[C]$ is extremal in $\overline{\mathrm{DNE}}(M)$.*

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[1]Usually, a line bundle L on a Kähler manifold (X, ω) is called nef if for every $\varepsilon > 0$, there exists a hermitian metric on L with curvature $\Theta \geq -\varepsilon\omega$. This is stronger than algebraic nefness, but these two notions coincide when X is a smooth projective variety.

[2]By Hodge theory, this coincides with the set of real Hodge classes $H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$.

[3]An element x in a closed cone C is called *extremal* if whenever $x = y + z$ for $y, z \in C$, then y and z are multiples of x.

When X is a smooth projective threefold, Mori gave a classification of the extremal rays in

$$\overline{\mathrm{NE}}(X)_{K_X<0} := \overline{\{[C] \in \mathrm{NE}(X) \mid (K_X, C) < 0\}}$$

and showed that they are all generated by rational curves. We define the *minimal homology class* in an extremal ray R the class of a rational curve $[C] \in R$ such that $(-K_X, C)$ is minimal. Ruan used his main criterion to prove that extremal curves do not disappear under symplectic diffeomorphism for projective threefolds by verifying that a minimal homology class $[C]$ in each extremal ray of $\overline{\mathrm{NE}}(X)_{K_X<0}$ has a nonzero genus zero Gromov-Witten invariant $\mathrm{GW}_{0,[C]}$. Since the canonical class is also a symplectic invariant, *i.e.* $\phi^*c_1(K_Y) = c_1(K_X)$ (indeed, $c_1(K_X)$ is determined by any ω-tamed almost complex structure which form a connected set) and since genus zero Gromov-Witten invariants can be calculated by doing intersection theory on the moduli space of rational curves [LT99] [Sie99], we conclude that there exists a rational curve $C' \subset Y$ such that $\phi^*[C'] = [C]$ and $(K_Y, C') = (\phi^*K_Y, \phi^*[C']) = (K_X, C) < 0$.

Back to the Kähler case, since the projective case was verified by Ruan, it suffices to prove Theorem 1.1 for non-algebraic compact Kähler threefolds. Recently, Höring and Peternell generalized Mori's cone Theorem for compact Kähler manifolds [HP13].

Theorem 1.3 (Höring, Peternell). *Let X be a compact Kähler manifold. There exists a countable set $(C_i)_{i \in I}$ of rational curves on X such that*

$$0 < -K_X \cdot C_i \le \dim(X) + 1$$

and

$$\overline{\mathrm{NE}}(X) = \overline{\mathrm{NE}}(X)_{K_X \ge 0} + \sum_{i \in I} \mathbb{R}^+[C_i]$$

where the $\mathbb{R}^+[C_i]$ are all the distinct extremal rays of $\overline{\mathrm{NE}}(X)$ that meet $N_1(X)_{K_X<0}$. These rays are locally discrete in that half-space.

This Theorem together with the classification list of non-splitting families of rational curves on Kähler threefolds given by Campana and Peternell [CP97] allow us to follow Ruan's method to prove Theorem 1.1 in the Kähler context.

This note is organized as follows. In section 2, we exhibit the classification list of non-splitting families of rational curves on Kähler threefolds. In section 3 we recall some elementary properties of Gromov-Witten invariants which we will use. Finally, we prove Theorem 1.1 in the last section.

If Z is an analytic (sub)space, $[Z]$ will denote interchangeably its fundamental homology class and its Poincaré dual cohomology class throughout this note.

2. Classification of non-splitting families of rational curves

Let X be a compact Kähler threefold. A family of rational curves $(C_t)_{t \in T}$ in X is called *non-splitting*, if T is compact and irreducible and every curve C_t is irreducible. Every extremal rational curve C such that $(K_X, C) < 0$ determines a non-splitting family of rational curves $(C_t)_{t \in T}$ [CP97], and $\dim T = -(K_X, C)$ by definition. One can classify these families according to $-(K_X, C) \in \{1, 2, 3, 4\}$ and this was done by Campana and Peternell [CP97]. Since we are mainly interested in non-algebraic compact threefolds, we shall exclude projective varieties from the classification list.

Theorem 2.1. *Let X be a non-algebraic compact Kähler threefold and (C_t) a non-splitting family of rational curves. Then either $(K_X, C_t) = -2$ or -1. Moreover,*

(1) *If $(K_X, C_t) = -2$, then*
  (a) *if C_t fills up a surface $S \subset X$, then $S \simeq \mathbb{P}^2$ with normal bundle $N_{S|X} = O(-1)$;*
  (b) *if C_t fills up X, then X is a $\mathbb{P}^1$-bundle over a surface, the C_t being fibers.*
(2) *If $(K_X, C_t) = -1$, then C_t fills up a surface $S \subset X$.*
  (a) *If S is normal, then one of the following holds:*
      (i) *$S \simeq \mathbb{P}^2$ with $N_{S|X} = O(-2)$;*
      (ii) *$S \simeq \mathbb{P}^1 \times \mathbb{P}^1$ with $N_{S|X} = O(-1, -1)$;*
      (iii) *S is a quadric cone with $N_{S|X} = O(-1)$;*
      (iv) *S is a ruled surface over a smooth curve C, and X is the blow-up of a smooth threefold along C.*
  (b) *If S is non-normal, then $\kappa(X) < 0$, $N_{S|X} = O_S$, and the normalization of S is either $\mathbb{P}^2$ or a ruled surface.*

3. Preliminaries on Gromov-Witten invariants

Let $A \in H_2(X, \mathbb{Z})$. We denote by $\overline{\mathcal{M}}_{A,n}(X)$ (or $\overline{\mathcal{M}}_{A,n}$) the moduli space of stable maps from curves of genus 0 to X with n marked points, whose homology class of its image is equal to A. When there is no marked point, this moduli space is simply denoted by $\overline{\mathcal{M}}_A$. It is a proper Deligne-Mumford stack. Since X is a complex threefold, we recall that the *expected* (or virtual) dimension of $\overline{\mathcal{M}}_{A,n}$

$$d_{\exp} = n - \int_A c_1(K_X),$$

and $\overline{\mathcal{M}}_{A,n}$ carries a virtual fundamental class of expected dimension $2d_{\exp}$

$$[\overline{\mathcal{M}}_{A,n}]^{\mathrm{vir}} \in H_{2d_{\exp}}(\overline{\mathcal{M}}_{A,n}, \mathbb{Q}).$$

Gromov-Witten invariants are defined by capping the cohomology classes against the virtual fundamental class of the space of stable maps. More precisely, given cohomology classes $A_1, \ldots, A_n$ in $H^*(X, \mathbb{Q})$, the corresponding genus zero Gromov-Witten invariant is defined by:

$$GW_{0,A}(A_1, \ldots, A_n) := \int_{[\overline{\mathcal{M}}_{A,n}]^{\mathrm{vir}}} e_1^*(A_1) \smile \cdots \smile e_n^*(A_n),$$

where e_i denotes the evaluation map with respect to the ith marked point

$$e_i : \overline{\mathcal{M}}_{A,n} \longrightarrow X$$
$$(\Sigma; p_1, \ldots, p_n; f) \longmapsto f(p_i).$$

When there is no obstruction on $\overline{\mathcal{M}}_{A,n}$, virtual fundamental class coincides with ordinary fundamental class. For instance, this happens when $H^1(C, f^*TX) = 0$ for all stable maps $f : C \to X$ [FP97]. When $\overline{\mathcal{M}}_{A,n}$ is non-singular, the canonical obstruction sheaf $\mathcal{T}^2$ is a vector bundle, so the virtual fundamental class is the Euler class of $\mathcal{T}^2$:

$$[\overline{\mathcal{M}}_{A,n}]^{\mathrm{vir}} = e(\mathcal{T}^2) \cap [\overline{\mathcal{M}}_{A,n}];$$

see [HKK$^+$03] for more details.

For each embedded curve $f : C \to X$, the pushfoward by f of the fundamental class of C and its Poincaré dual in $H^2(X, \mathbb{Z})$ are all denoted by $[C]$ in this note.

Remark 3.1. This remark is due to Claire Voisin. Theorem 1.1 is true for compact Kähler surfaces for simple reasons. Let X and Y be compact Kähler surfaces. If K_X is not nef, then either X contains a (-1)-curve C, or $\kappa(X) < 0$. If $f : C \to X$ is an embedded (-1)-curve, then $H^1(C, f^*TX) = H^1(\mathbb{P}^1, \mathcal{O}(-1)) = 0$, so there is no obstruction on $\overline{\mathcal{M}}_{[C]}$. The moduli space is zero dimensional, and $GW_{[C]} = 1$. If $\kappa(X) < 0$, then by classification of minimal compact Kähler surfaces, X is uniruled. Since uniruledness is preserved by deformed symplectic diffeomorphism[Kol98, Rua99], Y is also uniruled. In either case, K_Y is not nef.

4. Non-nef canonical bundle and symplectic geometry

This section is devoted to the proof of Theorem 1.1.

Proof of Theorem 1.1. Suppose K_X is not nef, and C is an extremal rational curve such that $(K_X, C) < 0$ whose existence is given by Höring-Peternell's cone Theorem 1.3. The case where X is projective was treated in [Rua99]. As before, we thus assume that X is non-algebraic. The curve C determines a non-splitting family of rational curves (C_t) which is classified in Theorem 2.1. Since the canonical class $c_1(K_X)$ is a symplectic invariant, it suffices to show that some genus zero Gromov-Witten invariants $GW_{0,[C]}(\cdots)$ is non zero for all cases listed in Theorem 2.1.

Lemma 4.1. *For the cases* (1.*a*), (1.*b*), (2.*a.ii*), (2.*a.iii*) *and* (2.*a.iv*), *the moduli space* $\overline{\mathcal{M}}_{[C]}$ *is unobstructed. Furthermore,* $GW_{0,[C]} \neq 0$.

Proof. Since the normal bundles $N_{C|X}$ in the cases (1.*a*), (1.*b*), (2.*a.ii*), (2.*a.iii*) and (2.*a.iv*) are

$$\mathcal{O}(1) \oplus \mathcal{O}(-1), \mathcal{O} \oplus \mathcal{O}, \mathcal{O} \oplus \mathcal{O}(-1), \mathcal{O} \oplus \mathcal{O}(-1), \text{ and } \mathcal{O} \oplus \mathcal{O}(-1)$$

respectively, one has $H^1(C, f^*T_X) = H^1(C, N_{C|X}) = 0$, so the moduli space is unobstructed.

For (1.*a*), since $N_{S|X} = \mathcal{O}(-1)$, we can perturb C to get a genus 0 sphere C' such that C' meets transversally S at a point x with intersection number (-1). Hence $GW_{0,[C]}([C], [C]) = 1$. For (1.*b*), it is clear that $GW_{0,[C]}([x]) = 1$ with $x \in X$. In the cases (2.*a.ii*), (2.*a.iii*) and (2.*a.iv*), one has $GW_{0,[C]}([C]) = [C] \cdot [S] = -1$ by adjunction formula.

$\square$

Remark 4.2. We can also consider Lemma 4.1 for the cases (1.*b*), (2.*a.ii*), (2.*a.iii*) and (2.*a.iv*) as a consequence of the following Lemma, which slightly generalizes Lemma 5.3 in [Rua99] in the context of Kähler geometry:

Lemma 4.3. *Suppose that the deformation of C is unobstructed. Let $e : \overline{\mathcal{M}}_{[C],1} \to X$ be the evaluation map. If* $\dim \overline{\mathcal{M}}_{[C]} \leq 2$ *and*

$$\dim e(\overline{\mathcal{M}}_{[C],1}) = \dim \overline{\mathcal{M}}_{[C],1},$$

then $\mathrm{GW}_{0,[C]} \neq 0$.

Proof. Since $\dim e(\overline{\mathcal{M}}_{[C],1}) = \dim \overline{\mathcal{M}}_{[C],1}$, one has

$$e_*[\overline{\mathcal{M}}_{[C],1}] = k[e(\overline{\mathcal{M}}_{[C],1})]$$

where $k = \deg e \neq 0$.

If $\dim \overline{\mathcal{M}}_{[C]} = 2$ and $x \in X$ is a point in general position, then

$$\mathrm{GW}_{0,[C]}(x) = e^*[x] \smile [\overline{\mathcal{M}}_{[C],1}] = k \neq 0$$

by projection formula.

If $\dim \overline{\mathcal{M}}_{[C]} = 1$, again by projection formula, one has

$$\mathrm{GW}_{0,[C]}(\alpha) = k \int_{e(\overline{\mathcal{M}}_{[C],1})} \alpha,$$

for $\alpha \in H^4(X, \mathbb{Q})$. Since $H^4(X, \mathbb{Q})$ is dense in $H^4(X, \mathbb{R})$, we can choose $\alpha \in H^4(X, \mathbb{Q})$ sufficiently close to $\omega \wedge \omega$, hence

$$\mathrm{GW}_{0,[C]}(\alpha) = k \int_{e(\overline{\mathcal{M}}_{[C],1})} \alpha \neq 0.$$

$\square$

For (2.*a.i*), one has $H^1(C, f^*T_X) = H^1(C, N_{C|X}) = \mathbb{C}$, so the deformation of C is obstructed. Since $\overline{\mathcal{M}}_{[C]} \simeq G(2,3) \simeq \mathbb{P}^2$ is non-singular, the virtual fundamental class

$$[\overline{\mathcal{M}}_{A,n}]^{\mathrm{vir}} = e(\mathcal{T}^2) \cap [\overline{\mathcal{M}}_{A,n}],$$

where $e(\mathcal{T}^2)$ is the Euler class of the obstruction bundle $\mathcal{T}^2$ and was first computed by Ruan [Rua93].

Lemma 4.4. $e(\mathcal{T}^2) = -\sigma_1$ *where σ_1 is the Schubert cycle which represent all the lines in $\mathbb{P}^2$ passing through a point in general position. Moreover,* $\mathrm{GW}_{0,[C]}([C]) = [C] \cdot [S] = -2$.

Proof. cf. [Rua93], section 5. $\square$

It remains the case (2.*b*), where C_t fills up a non-normal surface S. We denote by $\nu : \tilde{S} \to S$ the normalization of S.

Lemma 4.5. *If $\tilde{S}$ is a ruled surface, then* $\dim \overline{\mathcal{M}}_{[C]} = 1$, *and* $\mathrm{GW}_{0,[C]} \neq 0$.

Proof. First we note that $\dim \overline{\mathcal{M}}_{[C]} \geq -(K_X, C) = 1$. Since ν is finite and $\overline{\mathcal{M}}_{[C]}(X) = \overline{\mathcal{M}}_{[C]}(S)$, the normalization of $\overline{\mathcal{M}}_{[C]}(X)$ is included in the moduli space $\overline{\mathcal{M}}_{\nu^*[C]}(\tilde{S})$. It follows that $\dim \overline{\mathcal{M}}_{[C]} \leq \overline{\mathcal{M}}_{\nu^*[C]}(\tilde{S}) \leq 1$ because $\tilde{S}$ is a ruled surface.

As in the proof of Lemma 4.3, since $\dim e(\overline{\mathcal{M}}_{[C],1}) = \dim S = 2 = \dim \overline{\mathcal{M}}_{[C],1}$ and $\dim[\overline{\mathcal{M}}_{[C],1}] = \dim[\overline{\mathcal{M}}_{[C],1}]^{\mathrm{vir}}$, one has

$$e_*[\overline{\mathcal{M}}_{[C],1}]^{\mathrm{vir}} = k[e(\overline{\mathcal{M}}_{[C],1})]$$

for some non-zero integer k. Hence $\mathrm{GW}_{0,[C]}(\alpha) \neq 0$ for $\alpha \in H^4(X, \mathbb{Q})$ sufficiently close to $\omega \wedge \omega$.

$\square$

From now on, we concentrate on the last remaining case in (2.*b*), that is the case where the normalization of S is $\mathbb{P}^2$.

Lemma 4.6. *If $H^1(X, \mathcal{O}_X) = 0$, then X is a fiber space over $\mathbb{P}^1$ and S is contracted to a point.*

Proof. Consider the following short exact sequence:

$$0 \longrightarrow H^0(X, \mathcal{O}_X) \longrightarrow H^0(X, \mathcal{O}_X(S)) \longrightarrow H^0(X, \mathcal{O}_S(S)) \longrightarrow 0.$$

Thus it is clear that $\dim H^0(X, O_X(S)) \geq 2$. The morphism $X \to \mathbb{P}^1$ defined by the base-point-free linear system $|O_X(S)|$ does the work. $\qquad \square$

Lemma 4.7. *If $H^1(X, O_X) \neq 0$ and $\tilde{S} = \mathbb{P}^2$, then X is a fiber space over a smooth curve and S is contracted to a point.*

Proof. Since $H^1(X, O_X) \neq 0$, the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is non constant, so $\dim \mathrm{Im}\,\alpha \geq 1$. If α is generically finite, then the pullback of a general holomorphic 3-form is non-zero in $H^0(X, K_X)$, which contradicts the fact that $\kappa(X) < 0$.

Now we assume that $\dim \mathrm{Im}\,\alpha \leq 2$. Let ω_0 be a Kähler form on $\mathrm{Alb}(X)$ and $Q(\beta, \gamma) := \int_X \beta \wedge \gamma \wedge \omega$ be the intersection form on $H^{1,1}_{\mathbb{R}}(X) := H^{1,1}(X, \mathbb{C}) \cap H^2(X, \mathbb{R})$ determined by ω. The signature of Q is $(1, \dim H^{1,1}_{\mathbb{R}}(X) - 1)$ by Hodge index Theorem [Voi02, Theorem 6.2.3]. Note that since there is no non constant map from $\mathbb{P}^2 = \tilde{S}$ to any torus, α contracts S to a point, so $Q(\alpha^*\omega_0, [S]) = 0$. Since $\dim \mathrm{Im}\,\alpha \leq 2$, one has $Q(\alpha^*\omega_0, \alpha^*\omega_0) \geq 0$, it follows that Q is negative on the orthogonal complement in $H^{1,1}_{\mathbb{R}}(X)$ of the line determined by $[\alpha^*\omega_0]$. We then deduce from the fact that $Q(\alpha^*\omega_0, [S]) = 0$ and $O_S(S) = O_S$ (so $Q([S], [S]) = 0$) that $[S]$ is proportional to $[\alpha^*\omega_0]$. Hence $\dim \mathrm{Im}\,\alpha = 1$. $\qquad \square$

By Lemmata 4.6 and 4.7, if $\tilde{S} = \mathbb{P}^2$, then X is a fiber space over a smooth curve $\pi : X \to B$ such that S is a connected component of a fiber F. Assume that F is the disjoint union of S and S'. Since $Q([S], [S]) = 0$ and $Q([F], [F]) = 0$, one would have $Q([S], [F]) \neq 0$ (otherwise $[S]$ would be proportional to F again by Hodge index Theorem). Thus $Q([S], [S']) \neq 0$, which yields a contradiction. Hence S is a fiber of π.

Since $\nu^* K_S = O_{\mathbb{P}^2}(-1)$, one has $H^0(S, nK_S) = 0$ for all $n > 0$, so $\kappa(S) = -\infty$. The same result extends by semi-continuity to all fibers of points lie in a Zariski neighborhood of $\pi(S)$, *i.e.*, there is a non empty Zariski open $U \subset B$ such that $\kappa(S_t) = -\infty$ for all $t \in U$ where $S_t := \pi^{-1}(t)$. These S_t are all uniruled, so X is also uniruled. Since uniruledness is symplectic invariant [Kol98, Rua99], we conclude that Y is also uniruled, and the proof is done.

$\qquad \square$

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